1.0

1.1

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⑥ CONSERVATION LAWS WITH DISSIPATION

by

⑩ C. M. Dafermos
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island   02912

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1. Introduction

The conservation laws of isothermal, isentropic or adiabatic thermoelasticity, in all their standard variants (Lagrangean or Euclidean formulation, solids or fluids, one or several space dimensions, etc.), lead to systems of quasilinear hyperbolic equations. A feature of such systems is that the Cauchy problem does not have global smooth solutions, even when the initial data are very smooth, due to the formation of shock waves. However, global solutions exist in the class of functions of bounded variation, in the sense of Tonelli-Cesari [8].

When the material is viscous and/or heat may diffuse, dissipative mechanisms emerge in the system of conservation laws, which manifest themselves through the appearance of "parabolic" terms or "memory" terms. The same phenomenon also arises in the context of the theory of chemically reacting media with dissipation induced by diffusion.

A dissipative mechanism may affect, in general, the asymptotic behavior as well as the smoothness of solutions. Ranked according to effectiveness, dissipative mechanisms may be classifed into (a) those which are so powerful as to smoothen out even rough initial data, always yielding smooth solutions; (b) those that preserve the smoothness of smooth initial data but are incapable of smoothening rough initial data; (c) those that preserve the smoothness of smooth and "small" initial data but cannot prevent the breaking of smooth waves of large amplitude; (d) those that are not capable to prevent even the breaking of smooth waves of small amplitude.

From the viewpoint of analysis as well as continuum physics it would be useful to classify the conservation laws for the

standard material classes into one of the above categories. Although a number of scattered results in this direction are already available, there are still many unanswered questions and, more importantly, there has been no attempt to place the existing information into a general framework.

In this lecture, I shall outline a research program towards understanding the role of dissipation and estimating its effectiveness. I will exhibit a (far from exhaustive) list of representative examples and I will discuss what has been established and what has been conjectured, for each case.

2. Complete Parabolic Damping

The simplest example of an equation with complete parabolic damping is provided by

$$(2.1) \qquad u_t + f(u)_x = u_{xx},$$

($f(u)$ nonlinear and smooth) whose behavior is to be contrasted to that of the hyperbolic conservation law

$$(2.2) \qquad u_t + f(u)_x = 0.$$

The Cauchy problem for (2.2), with initial data $u(x,0)$, of bounded variation, admits a solution in the class BV of functions of bounded variation, in the sense of Tonelli-Cesari. No gain would be made by assuming that $u(x,0)$ is smoother, even analytic! In contrast, the initial-value problem for (2.1) with rough, just

bounded measurable, initial data always has a smooth solution. This type of behavior is, of course, due to the power of parabolic damping and characterizes a broad class of systems of equations in the form

(2.3)
$$\underset{\sim}{w}_t + \underset{\sim}{f}(\underset{\sim}{w})_x = A\underset{\sim}{w}_{xx}$$

where A is a positive definite matrix and $\underset{\sim}{f}$ satisfies certain technical smoothness and growth assumptions.

3. Incomplete Parabolic Damping

In this section we will discuss the representative systems

(3.1)
$$\begin{cases} u_t - v_x = 0 \\ v_t + p(u)_x = v_{xx} \end{cases}$$

(3.2)
$$\begin{cases} u_t - v_x = 0 \\ v_t + p(u,\theta)_x = v_{xx} \\ [e(u,\theta) + \frac{v^2}{2}]_t + [p(u,\theta)v]_x - [vv_x]_x = \theta_{xx} \end{cases}$$

(3.3)
$$\begin{cases} u_t - v_x = 0 \\ v_t + p(u,\theta)_x = 0 \\ [e(u,\theta) + \frac{v^2}{2}]_t + [p(u,\theta)v]_x = \theta_{xx}. \end{cases}$$

(3.1) are the equations of motion of one-dimensional viscoelastic

materials of the rate type, while (3.2) and (3.3) express the con-
servation laws of momentum and energy of one-dimensional thermo-
viscoelastic and thermoelastic materials, respectively.

The physically natural assumptions are

(3.4) $p_u(u,\theta) < 0$, $e_\theta(u,\theta) > 0$, $e_u(u,\theta) = \theta^2 [\frac{p(u,\theta)}{\theta}]_\theta$.

(3.1) should be contrasted to the system of equations of motion
of one-dimensional elastic materials,

$$
(3.5) \qquad
\begin{cases}
u_t - v_x = 0 \\[2mm]
v_t + p(u)_x = 0,
\end{cases}
$$

while (3.2) and (3.3) should be contrasted to the conservation laws
for adiabatic processes in one-dimensional thermoelasticity:

$$
(3.6) \qquad
\begin{cases}
u_t - v_x = 0 \\[2mm]
v_t + p(u,\theta)_x = 0 \\[2mm]
[e(u,\theta) + \frac{v^2}{2}]_t + [p(u,\theta)v]_x = 0.
\end{cases}
$$

(3.5) and (3.6) are typical examples of systems of hyperbolic
conservation laws.

System (3.1) can be written in the form (1.3) where, however,
A is only positive semidefinite (whence the term "incomplete
parabolic damping"). Intensive investigation for System (3.1)
(e.g. [1,3,9,10]) has generated a chain of existence theorems

revealing several function classes whose smoothness is preserved by solutions. Typical examples of such classes are $L^\infty \times L^2$ (cf. [1]) and $C^{2+\alpha} \times C^{2+\alpha}$ (cf. [3]). However, an important link is missing from the chain. Indeed, since the Cauchy problem for (3.5), with initial data of small total variation, has global solutions of class BV, it is natural to expect the same from (3.1). This conjecture, however, has not yet been verified. The difficulty lies in that the existence theory for (3.5) has been established by means of Glimm's scheme [8] which hinges on the explicit construction of solutions to the Riemann problem for systems of hyperbolic conservation laws. Numerical analysts [2] have applied Glimm type methods (in combination with fractional steps or by modifying the Riemann problem) to systems containing "parabolic" terms with excellent results. Nevertheless, the theoretical justification of this approach is still lacking.

Solutions of (3.1) of class BV have different geometric structure than corresponding solutions of (3.5). For functions (u,v) of class BV, first derivatives u_t, u_x, v_t, v_x are locally finite Borel measures. When such a pair of functions is a solution to (3.5), the set of points of jump discontinuity is (cf. [7]) the countable union of C^1 curves (shocks) with slope

$$(3.7) \qquad s = \pm\left\{-\frac{[p(u)]}{[u]}\right\}^{1/2}.$$

The jumps of u and v across shocks are controlled by the Rankine-Hugoniot conditions

$$(3.8) \quad \begin{cases} s[u] + [v] = 0 \\ \\ s[v] - [p(u)] = 0. \end{cases}$$

On the other hand, if $(u,v) \in BV$ is a solution to (3.1), it follows from $(3.1)_2$ that v_{xx} is also a locally finite Borel measure. Thus, v cannot sustain any jump discontinuity and, as seen from $(3.1)_1$, u may only be discontinuous along stationary lines, $s = 0$. Through explicit construction of particular solutions of (3.1) one shows that such singularities may indeed occur. Consequently, in contrast to the complete parabolic damping of Section 2, viscous damping is incapable of smoothening rough initial data.

My conjecture is that System (3.2) exhibits exactly the same behavior as (3.1). A research program towards verifying this conjecture is in progress.

In System (3.3) damping is weaker than in (3.1) or (3.2). Slemrod [14] has shown that global smooth solutions exist provided that the initial data are both smooth and "small". My conjecture is that smooth waves of large amplitude break. Since the Cauchy problem for (3.6) has a global solution of class BV when the initial data have small total variation [11], it is natural to expect similar behavior from (3.3). However, this conjecture has not been established yet.

Jump discontinuities of BV solutions of (3.6) occur across forward or backward shocks, propagating with speeds

$$(3.9) \qquad s = \pm \left\{ - \frac{[p(u,\theta)]}{[u]} \right\}^{1/2},$$

or across contact discontinuities which are stationary, $s = 0$, and across which u and θ may jump but v and p are continuous. On the other hand, if $(u,v,\theta) \in BV$ is a solution of (3.3), θ_{xx} is also a locally finite Borel measure. It follows that θ cannot sustain jump discontinuities. Forward and backward shocks, propagating at speeds shown in (3.9), are still possible. Contact discontinuities, however, are ruled out since u alone cannot jump when all v, θ, and $p(u,\theta)$ are continuous.

4. Viscous Damping Induced by Memory Effects

In this section, we discuss dissipation mechanisms induced by viscosity in simple materials with fading memory (e.g. [16]). As representative examples consider

$$(4.1) \qquad \begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x - \int_{-\infty}^{t} a(t-\tau)\phi(u)_x d\tau = 0 \end{cases}$$

$$(4.2) \qquad \begin{cases} u_t - v_x + \mu \int_{-\infty}^{t} e^{-\mu(t-\tau)} v_x d\tau = 0 \\ v_t - \sigma(u)_x - \int_{-\infty}^{t} a(t-\tau)\phi(u)_x d\tau = 0. \end{cases}$$

System (4.1) is a model of the equation of motion of non-linear, one dimensional viscoelastic materials of the Boltzmann type while (4.2)

is the equation of motion of another model material (v is velocity in both (4.1) and (4.2) while u is a strain in (4.1) and an internal variable, function of the history of velocity gradient, in (4.2). The functions σ and ϕ are smooth and strictly increasing and the kernel $a(t)$ is a relaxation function.

Let us generally consider systems

$$(4.3) \qquad \underset{\sim}{w}_t + \underset{\sim}{f}(\underset{\sim}{w})_x + \int_{-\infty}^{t} A(t-\tau)\underset{\sim}{g}(\underset{\sim}{w})_x d\tau = 0$$

where $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are smooth maps from R^n to R^n and $A(t)$ is a smooth $n \times n$ matrix-valued relaxation kernel. We shall be looking for solutions near $\underset{\sim}{w} = \underset{\sim}{0}$ and we will be assuming that $\nabla \underset{\sim}{f}(\underset{\sim}{0})$ has n real distinct eigenvalues (strict hyperbolicity), none of them zero, in order to exclude the possibility of stationary waves that would overburden the memory term.

In systems of the above form, dissipation may be induced by the memory term. In order to expose the instantaneous component of damping, we define a new relaxation function

$$(4.4) \qquad B(t) = A(t)\nabla \underset{\sim}{g}(\underset{\sim}{0})[\nabla \underset{\sim}{f}(\underset{\sim}{0})]^{-1}$$

and write

$$(4.5) \qquad A(t)\underset{\sim}{g}(\underset{\sim}{w}) = B(t)\underset{\sim}{f}(\underset{\sim}{w}) + A(t)\underset{\sim}{h}(\underset{\sim}{w})$$

where

$$(4.6) \qquad \underset{\sim}{h}(\underset{\sim}{w}) = \underset{\sim}{g}(\underset{\sim}{w}) - \nabla \underset{\sim}{g}(\underset{\sim}{0})[\nabla \underset{\sim}{f}(\underset{\sim}{0})]^{-1} \underset{\sim}{f}(\underset{\sim}{w}).$$

Then (4.3) takes the form

$$(4.7) \qquad \underset{\sim}{w}_t + \underset{\sim}{f}(\underset{\sim}{w})_x + \int_{-\infty}^{t} B(t-\tau)\underset{\sim}{f}(\underset{\sim}{w})_x d\tau = - \int_{-\infty}^{t} A(t-\tau)\underset{\sim}{h}(\underset{\sim}{w})_x d\tau.$$

Next, we consider the resolvent kernel $K(t)$ of $B(t)$, i.e., the solution to the linear Volterra equation

$$(4.8) \qquad K(t) + \int_{0}^{t} B(t-\tau)K(\tau)d\tau = -B(t).$$

Forming the convolution of (4.7) with $K(t)$ and after a simple computation we arrive at

$$(4.9) \qquad \underset{\sim}{w}_t + \underset{\sim}{f}(\underset{\sim}{w})_x + K(0)\underset{\sim}{w}$$

$$= -\int_{-\infty}^{t} K'(t-\tau)\underset{\sim}{w}d\tau + \int_{-\infty}^{t} E(t-\tau)\underset{\sim}{h}(\underset{\sim}{w})_x d\tau$$

where

$$(4.10) \qquad E(t) = A(t) + \int_{0}^{t} K(t-\tau)A(\tau)d\tau.$$

Equation (4.9) is more convenient than the original form, (4.3). In the first place, if $B(0)$ is negative definite, then the term $K(0)\underset{\sim}{w}$ on the left-hand side of (4.9) induces instantaneous damping. Furthermore, the integral terms on the right-hand side of (4.9) are tame, the first one because it is linear in $\underset{\sim}{w}$ and does not involve

any derivatives and the second because it is "small", by virtue of $\nabla \underline{h}(\underline{0}) = \underline{0}$.

The damping in (4.9) is quite weak. Even so, in several special cases, including (4.1) with an appropriate relaxation kernel, it has been shown [4,5,12,13,15] that there exists a global smooth solution to the Cauchy problem provided that the initial data are both smooth and "small". On the other hand, when the initial data have small total variation, L. Hsiao and the author [6] have established, by means of a modification of the method of Glimm, the existence of a global solution of class BV. The strategy in [6] is to show that damping counterbalances the effect of the integral terms on the right-hand side of (4.9).

Clearly, a lot of work is still needed in order to complete the program of classification of conservation laws with dissipation.

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